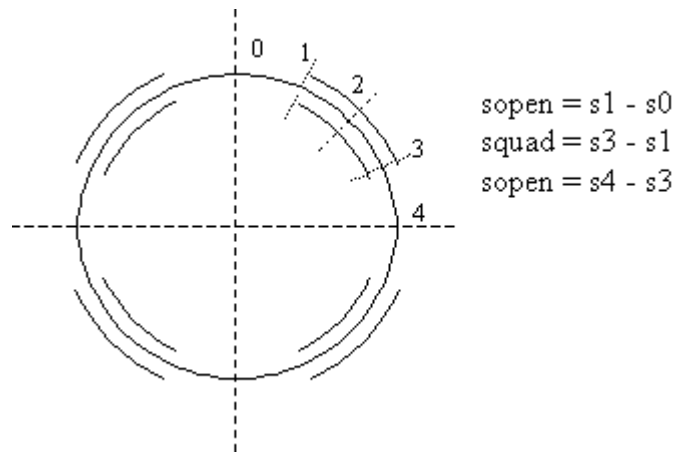


The beta function of the g-2 ring.

In this note vertical and horizontal beta functions for g-2 ring are calculated with the techniques of Courant and Snyder, Ann. Phys. 3, 1 (1958). These techniques are also carefully discussed in the text by Edwards and Syphers, Introduction to the Physics of High Energy Accelerators. The results of this note are compared to the results of g-2 note #77 and to the program IDA by Dr. Mark Barton of BNL. The calculation uses an average n value (0.134) which gives a CBO frequency of 465.5 kHz. The alpha, beta and psi functions are evaluated in approximately equal step sizes through out the ring. The calculation is done with the program MathCad, and this note is a reformated version of the MathCad document. Note that in MathCad “:=” denotes a variable or function assignment, and “=” denotes evaluation of an expression.

Define some physical constants. $c := 2.9979245810^8$ $a := 1.165920310^{-3}$

First define the ring geometry as shown in the figure below. The ring has four-fold symmetry. The ring is 7.112 m in radius. The quads are 39 degrees in azimuth. Find the arc lengths through the quad and the open (non-quad) sections. All values below are in meters. Note that sopen is one-half of the distance between the quads due to the definition of the period used below.



$$R := 7.112 \quad \text{arcquad} := 39 \quad \text{squad} := 2 \cdot \pi \cdot R \cdot \frac{39}{360} \quad \text{squad} = 4.8409848 \quad \text{sopen} := \frac{1}{2} \cdot \left(\frac{2 \cdot \pi \cdot R}{4} - \text{squad} \right) \quad \text{sopen} = 3.1652593 \quad \text{smag} := \text{sopen}$$

Check $4 \cdot \frac{(2 \cdot \text{sopen} + \text{squad})}{2 \cdot \pi \cdot R} = 1.0000000$

Average n value is 0.134. Find n value of focusing electrodes. $\text{naverage} := 0.13403$ $n := \text{naverage} \cdot \frac{360}{4 \cdot \text{arcquad}} \quad n = 0.3093046$

In linear approximation the equations for horizontal and vertical motions (the Kerst-Serber equations) are of the form $y(s)'' + K(s)y(s) = 0$, where s is the arc length and differentiation is with respect to s . Moreover, for a ring $K(s+C)=K(s)$, where C is the circumference of the ring. The functions y and y' represent either the horizontal (often denoted by x and x') or the vertical (often denoted by z and z') position and slope of a particle. K is related to the field strength and is constant within each section of the g-2 ring. We find the phase angle solution, $y(s)$, below. First we find a piecewise solution to the motion of a particle.

Each section of the lattice is represented by a 2 by 2 transport matrix

$$\begin{pmatrix} y \\ \frac{d}{ds}y \end{pmatrix}_2 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} y \\ \frac{d}{ds}y \end{pmatrix}_1$$

which takes the particle from position 1 to position 2. The matrix elements are simple functions of the arc length and the field strength. These functions are well known from charged particle optics. Matrix elements T_{11} and T_{22} are dimensionless, matrix element T_{12} has dimensions of length, and matrix element T_{21} has dimensions of inverse length. The matrix has unit determinant. For our application we need four matrices: O (M) for the vertical (horizontal) motion between the quads where the field strength is 0 (K_{mag}), and F_v (F_h) for the vertical (horizontal) motion in the quads where the field strength is K_v (K_h). Since the ring has four fold symmetry, the period of the lattice is one quarter of the circumference of the ring. Within a period the lattice in the vertical is OF_vO and the lattice in the horizontal is MF_hM . Matrix elements for each section are calculated below.

Define field strengths and phase advances within a section. Note that $\frac{n}{R^2} = 0.0061151$

$K_v := \frac{n}{R^2}$	$K_v = 0.0061151$	$\phi_v := \sqrt{K_v} \cdot s_{quad}$	$\phi_v = 0.3785604$
$K_h := \frac{1-n}{R^2}$	$K_h = 0.0136554$	$\phi_h := \sqrt{K_h} \cdot s_{quad}$	$\phi_h = 0.5656988$
$K_{mag} := \frac{1}{R^2}$	$K_{mag} = 0.0197704$	$\phi_{mag} := \sqrt{K_{mag}} \cdot s_{mag}$	$\phi_{mag} = 0.4450590$

Define matrices O , F_v , F_h , and M using results from first-order charged particle optics and verify that the matrices have unit determinant.

$O := \begin{pmatrix} 1 & s_{open} \\ 0 & 1 \end{pmatrix}$	$O = \begin{pmatrix} 1.0000000 & 3.1652593 \\ 0.0000000 & 1.0000000 \end{pmatrix}$	$ O = 1.0000000$
$F_v := \begin{pmatrix} \cos(\phi_v) & \frac{\sin(\phi_v)}{\sqrt{K_v}} \\ -\sqrt{K_v} \cdot \sin(\phi_v) & \cos(\phi_v) \end{pmatrix}$	$F_v = \begin{pmatrix} 0.9291977 & 4.7261852 \\ -0.0289011 & 0.9291977 \end{pmatrix}$	$ F_v = 1.0000000$

$$F_h := \begin{pmatrix} \cos(\phi_h) & \frac{\sin(\phi_h)}{\sqrt{K_h}} \\ -\sqrt{K_h} \cdot \sin(\phi_h) & \cos(\phi_h) \end{pmatrix} \quad F_h = \begin{pmatrix} 0.8442142 & 4.5868868 \\ -0.0626356 & 0.8442142 \end{pmatrix} \quad |F_h| = 1.0000000$$

$$M := \begin{pmatrix} \cos(\phi_{\text{mag}}) & \frac{\sin(\phi_{\text{mag}})}{\sqrt{K_{\text{mag}}}} \\ -\sqrt{K_{\text{mag}}} \cdot \sin(\phi_{\text{mag}}) & \cos(\phi_{\text{mag}}) \end{pmatrix} \quad M = \begin{pmatrix} 0.9025853 & 3.0617949 \\ -0.0605331 & 0.9025853 \end{pmatrix} \quad |M| = 1.0000000$$

Define matrices for vertical and horizontal periods, P_v and P_h , and the vertical and horizontal lattice, L_v and L_h , and verify unit determinant.

$$P_v := O \cdot F_v \cdot O \quad P_v = \begin{pmatrix} 0.8377183 & 10.3189325 \\ -0.0289011 & 0.8377183 \end{pmatrix} \quad |P_v| = 1.0000000$$

$$P_h := M \cdot F_h \cdot M \quad P_h = \begin{pmatrix} 0.1075756 & 7.8155959 \\ -0.1264686 & 0.1075756 \end{pmatrix} \quad |P_h| = 1.0000000$$

$$L_v := P_v \cdot P_v \cdot P_v \cdot P_v \quad L_v = \begin{pmatrix} -0.6743045 & 13.9535167 \\ -0.0390807 & -0.6743045 \end{pmatrix} \quad |L_v| = 1.0000000$$

$$L_h := P_h \cdot P_h \cdot P_h \cdot P_h \quad L_h = \begin{pmatrix} 0.9084913 & -3.2852315 \\ 0.0531602 & 0.9084913 \end{pmatrix} \quad |L_h| = 1.0000000$$

Next we find closed form solution to the equations of motion. Courant and Snyder show that the solution to the differential equation with the periodicity condition is of the form $y(s) = \sqrt{A \cdot \beta(s)} \cdot \cos(\psi(s) + \delta)$. A and δ are constants determined by the initial position and slope of the particle. The amplitude function, $\beta(s)$, gives the beam envelope. Note that $\sqrt{A \cdot \beta(s)}$ has units of length. Courant and Snyder show that β must satisfy the differential equation,

$$2 \cdot \beta \cdot \frac{d^2}{ds^2} \beta - \left(\frac{d}{ds} \beta \right)^2 + 4 \cdot K \cdot \beta^2 = 4 \text{ alternatively, } \frac{d^3}{ds^3} \beta + 4 \cdot K \cdot \frac{d}{ds} \beta + 2 \cdot \frac{d}{ds} K \cdot \beta = 0,$$

which has simple closed form solutions when K is a constant. Note that K has units of $1 / \text{length squared}$. They also define auxiliary functions

$$\alpha(s) = \frac{-1}{2} \cdot \left(\frac{d}{ds} \beta \right), \quad \gamma = \frac{1 + \alpha^2}{\beta}, \quad \text{and} \quad \mu = \psi(L_{v,h}) - \psi(0) = \int_0^{L_{v,h}} \frac{1}{\beta} ds.$$

which can be obtained from β . Note that α and ψ are dimensionless. μ is the phase advance in one turn.

The relations, $\frac{d}{ds}\alpha = K\beta - \gamma$, $\frac{d}{ds}\gamma = 2K\alpha$ and $\frac{d}{ds}\psi = \frac{1}{\beta}$ are also useful.

Closed form expressions for α , β , γ , and ψ as a function of s are given for each section of the ring below. The initial values for α , β , γ (integration constants) are found from matrix elements of L_v and L_h using Courant and Snyder equation 2.20

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\mu) + \alpha \cdot \sin(\mu) & \beta \cdot \sin(\mu) \\ -\gamma \cdot \sin(\mu) & \cos(\mu) - \alpha \cdot \sin(\mu) \end{pmatrix}$$

which enforces the fact that the solution must be periodic. Note α , β , and γ are used both to denote an integration constant and a function of arc length, s . μ (proportional to the tune) is a number.

Define parameters of the matrix and find μ for vertical and horizontal. $\mu / 2\pi = \nu$ is the tune.

$$av := Lv_{0,0} \quad bv := Lv_{0,1} \quad cv := Lv_{1,0} \quad dv := Lv_{1,1} \quad ah := Lh_{0,0} \quad bh := Lh_{0,1} \quad ch := Lh_{1,0} \quad dh := Lh_{1,1}$$

$$\mu v := \text{acos}\left(\frac{av + dv}{2}\right) \quad \mu v = 2.3108188 \quad \mu h := \text{acos}\left(\frac{ah + dh}{2}\right) \quad \mu h = 0.4311367 \quad \mu h \text{prime} := 2\pi - \mu h \quad \mu h \text{prime} = 5.8520486$$

Note that μ is only determined modulo 2π by this method. Also we must choose correct quadrant for inverse cosine function. The value $\mu h \text{prime}$ agrees with direct integration of ψ below.

Compare exact tune with simple formulas for weak focusing synchrotron.

$$\begin{aligned} \text{vertical tune is } \frac{\mu v}{2\pi} &= 0.3677782 & \sqrt{\text{naverage}} &= 0.3661038 & \frac{\frac{\mu v}{2\pi} - \sqrt{\text{naverage}}}{\frac{\mu v}{2\pi}} &= 0.0045528 \\ \text{horizontal tune is } \frac{\mu h \text{prime}}{2\pi} &= 0.9313825 & \sqrt{1 - \text{naverage}} &= 0.9305740 & \frac{\frac{\mu h \text{prime}}{2\pi} - \sqrt{1 - \text{naverage}}}{\frac{\mu h \text{prime}}{2\pi}} &= 0.0008680 \end{aligned}$$

The simple formulas are clearly very good.

Find integration constants (initial values) α , β , and γ

$$\text{alphav} := \frac{av - dv}{2 \cdot \sin(\mu v)} \quad \text{alphav} = 0.0000000 \quad \text{alphah} := \frac{ah - dh}{2 \cdot \sin(\mu h)} \quad \text{alphah} = 0.0000000$$

$$\text{betav} := \frac{bv}{\sin(\mu v)}$$

$$\text{betav} = 18.8955947$$

$$\text{betah} := \left| \frac{bh}{\sin(\mu h)} \right|$$

$$\text{betah} = 7.8612153$$

$$\text{gammav} := \frac{1 + \text{alphav}^2}{\text{betav}}$$

$$\text{gammav} = 0.0529224$$

$$\text{gammah} := \frac{1 + \text{alphah}^2}{\text{betah}}$$

$$\text{gammah} = 0.1272068$$

$$\text{Check unit determinant condition. } \text{gammav} \cdot \text{betav} - \text{alphav}^2 = 1.0000000 \quad \text{gammah} \cdot \text{betah} - \text{alphah}^2 = 1.0000000$$

Note that alpha is zero at the beginning of the open section for the vertical motion, and at the beginning of the magnet section for the horizontal motion. Since α is proportional to the derivative of β , the β function is at a maximum or at a minimum at these positions. α , β , and γ collectively are called the CLS parameters (L for Livingstone). The values found above are the integration constants used in calculation of the α , β , γ , and ψ functions.

Recall that the vertical lattice is OFvO and the horizontal lattice is MFhM. We will transport the particle first through the vertical O and horizontal M sections, then the vertical Fv and horizontal Fh sections, and finally through the last vertical O and horizontal M sections to obtain the transport through the entire period.

First transport the particle through the open section (vertical motion, non-quad region). Define the CLS functions for an open section. In the expressions below α_i and β_i are the initial values at the beginning of the element, and s is the path length through the element. It is straightforward to show that these functions satisfy the equations given above for β , α , γ , and ψ . Recall $K = 0$ here.

apply condition $K = 0$ to equation for beta	$\beta''' + 4K\beta' + 2K'\beta = 0 \Rightarrow \beta''' + 4K\beta' = 0 \Rightarrow \beta''' = 0$
solve for beta and evaluate initial beta	$\beta = a + bs + cs^2 \Rightarrow \beta_i = a \Rightarrow a = \beta_i$
solve for alpha and evaluate initial alpha	$\alpha = -\frac{1}{2}\beta' \Rightarrow \alpha = -\frac{1}{2}b - cs \Rightarrow \alpha_i = -\frac{1}{2}b \Rightarrow b = -2\alpha_i$
solve for gamma and evaluate initial gamma	$\gamma = K\beta - \alpha' \Rightarrow \gamma = -\alpha' = c \Rightarrow \gamma = c \Rightarrow c = \gamma = \frac{1 + \alpha_i^2}{\beta_i}$
Then	
beta	$\beta = \beta_i - 2\alpha_i s + \frac{1 + \alpha_i^2}{\beta_i} s^2$
alpha	$\alpha = \alpha_i - \frac{1 + \alpha_i^2}{\beta_i} s$
gamma	$\gamma = \frac{1 + \alpha_i^2}{\beta_i}$
initial psi	$\psi_i = 0$

The functions then for an open section are

$$\beta_{\text{open}}(\alpha_i, \beta_i, s) := \beta_i - 2 \cdot \alpha_i \cdot s + \left(\frac{1 + \alpha_i^2}{\beta_i} \right) s^2$$

$$\alpha_{\text{open}}(\alpha_i, \beta_i, s) := \alpha_i - \left(\frac{1 + \alpha_i^2}{\beta_i} \right) s$$

$$\gamma_{\text{open}}(\alpha_i, \beta_i, s) := \frac{1 + \alpha_i^2}{\beta_i}$$

$$\psi_{\text{open}}(\alpha_i, \beta_i, s) := \int_0^s \frac{1}{\beta_i - 2 \cdot \alpha_i \cdot \sigma + \left(\frac{1 + \alpha_i^2}{\beta_i} \right) \sigma^2} d\sigma$$

Evaluate the CLS parameters at the end of the open O section, using the above functions and CLS parameters, α_{v0} , β_{v0} , γ_{v0} and ψ_{v0} as initial values. Values at the end of the open section are α_{v1} , β_{v1} , γ_{v1} and ψ_{v1} . These values then are the initial values for the next section.

$$\alpha_{v0} := \text{alphav} \quad \alpha_{v0} = 0.0000000 \quad \beta_{v0} := \text{betav} \quad \beta_{v0} = 18.8955947 \quad \psi_{v0} := 0 \quad \psi_{v0} = 0.0000000$$

$$\alpha_{v1} := \alpha_{\text{open}}(\alpha_{v0}, \beta_{v0}, s_{\text{open}}) \quad \alpha_{v1} = -0.1675131 \quad \beta_{v1} := \beta_{\text{open}}(\alpha_{v0}, \beta_{v0}, s_{\text{open}}) \quad \beta_{v1} = 19.4258170 \quad \psi_{v1} := \psi_{\text{open}}(\alpha_{v0}, \beta_{v0}, s_{\text{open}}) \quad \psi_{v1} = 0.1659721$$

Next transport the particle through the magnet section M (horizontal motion, non-quad). Define the CLS functions for the magnet section. (With appropriate choice of the bend parameter, κ , these functions can also be used for the Fv section, describing vertical motion in the quads, and the Fh section, describing horizontal motion in the quads where there is also a dipole component.) It is straightforward to show that these functions satisfy the equations given above for β , α , γ , and ψ . Recall $\kappa > 0$ here. If $\kappa < 0$ (not in a weak focusing ring), we would have cosh and sinh functions.

apply condition K = constant to equation for beta	$\beta''' + 4K\beta' + 2K'\beta = 0 \Rightarrow \beta''' + 4K\beta' = 0$
solve for beta and evaluate initial beta	$\beta = a + b \cos \sqrt{4K}s + c \sin \sqrt{4K}s \Rightarrow \beta_i = a + b \Rightarrow a + b = \beta_i$
solve for alpha and evaluate initial alpha	$\alpha = -\frac{1}{2}\beta' \Rightarrow \alpha = b\sqrt{K} \sin \sqrt{4K}s - c\sqrt{K} \cos \sqrt{4K}s$ $\Rightarrow \alpha_i = -c\sqrt{K} \Rightarrow c = -\frac{\alpha_i}{\sqrt{K}}$
solve for alphaprime	$\alpha' = 2bK \cos \sqrt{4K}s + 2cK \sin \sqrt{4K}s$
solve for gamma and evaluate initial gamma	$\gamma = K\beta - \alpha' \Rightarrow \gamma = K(a + b \cos \sqrt{4K}s + c \sin \sqrt{4K}s) - (2bK \cos \sqrt{4K}s + 2cK \sin \sqrt{4K}s)$ $\Rightarrow \gamma_i = K(a - b) \Rightarrow a - b = \frac{\gamma_i}{K}$
Then	
solve for a	$a = \frac{1}{2}\beta_i + \frac{1}{2} \frac{1 + \alpha_i^2}{K\beta_i}$
solve for b	$b = \frac{1}{2}\beta_i - \frac{1}{2} \frac{1 + \alpha_i^2}{K\beta_i}$
solve for c	$c = -\frac{\alpha_i}{\sqrt{K}}$
beta	$\beta = \left(\frac{1}{2}\beta_i + \frac{1}{2} \frac{1 + \alpha_i^2}{K\beta_i} \right) + \left(\frac{1}{2}\beta_i - \frac{1}{2} \frac{1 + \alpha_i^2}{K\beta_i} \right) \cos \sqrt{4K}s - \frac{\alpha_i}{\sqrt{K}} \sin \sqrt{4K}s$
alpha	$\alpha = \left(\frac{1}{2}\beta_i - \frac{1}{2} \frac{1 + \alpha_i^2}{K\beta_i} \right) \sqrt{K} \sin \sqrt{4K}s + \alpha_i \cos \sqrt{4K}s$
gamma	$\gamma = K\beta - \alpha'$
initial psi	$\psi_i = 0$

The functions then for a section with strength κ are

$$\beta_{\text{bend}}(\alpha_i, \beta_i, \kappa, s) := \frac{1}{2} \left(\beta_i + \frac{1 + \alpha_i^2}{\kappa \cdot \beta_i} \right) + \frac{1}{2} \left(\beta_i - \frac{1 + \alpha_i^2}{\kappa \cdot \beta_i} \right) \cos(\sqrt{4 \cdot \kappa} \cdot s) - \frac{\alpha_i}{\sqrt{\kappa}} \cdot \sin(\sqrt{4 \cdot \kappa} \cdot s)$$

$$\alpha_{\text{bend}}(\alpha_i, \beta_i, \kappa, s) := \frac{1}{2} \cdot \left(\beta_i - \frac{1 + \alpha_i^2}{\kappa \cdot \beta_i} \right) \sqrt{\kappa} \cdot \sin(\sqrt{4 \cdot \kappa} \cdot s) + \alpha_i \cdot \cos(\sqrt{4 \cdot \kappa} \cdot s)$$

$$\psi_{\text{bend}}(\alpha_i, \beta_i, \kappa, s) := \int_0^s \frac{1}{\left[\frac{1}{2} \cdot \left(\beta_i + \frac{1 + \alpha_i^2}{\kappa \cdot \beta_i} \right) + \frac{1}{2} \cdot \left(\beta_i - \frac{1 + \alpha_i^2}{\kappa \cdot \beta_i} \right) \cdot \cos(\sqrt{4 \cdot \kappa} \cdot \sigma) - \frac{\alpha_i}{\sqrt{\kappa}} \cdot \sin(\sqrt{4 \cdot \kappa} \cdot \sigma) \right]} d\sigma$$

Evaluate the CLS parameters at the end of the magnet section using the above functions and CLS parameters, α_{h0} , β_{h0} , γ_{h0} and ψ_{h0} as initial values. Values at the end of the magnet section are α_{h1} , β_{h1} , γ_{h1} and ψ_{h1} . These values are the initial values for the next section.

$$\alpha_{h0} := \text{alphah} \quad \alpha_{h0} = 0.0000000 \quad \beta_{h0} := \text{betah} \quad \beta_{h0} = 7.8612153 \quad \psi_{h0} := 0 \quad \psi_{h0} = 0.0000000$$

$$\alpha_{h1} := \alpha_{\text{bend}}(\alpha_{h0}, \beta_{h0}, K_{\text{mag}}, s_{\text{mag}}) \quad \alpha_{h1} = 0.0779674$$

$$\beta_{h1} := \beta_{\text{bend}}(\alpha_{h0}, \beta_{h0}, K_{\text{mag}}, s_{\text{mag}}) \quad \beta_{h1} = 7.5967305$$

$$\psi_{h1} := \psi_{\text{bend}}(\alpha_{h0}, \beta_{h0}, K_{\text{mag}}, s_{\text{mag}}) \quad \psi_{h1} = 0.4073778$$

Next evaluate the CLS parameters in the middle of the quad section for diagnostic purposes. Note that ψ_{v1} and ψ_{h1} are added to the integrals to obtain the integral phase shift.

Vertical motion

$$\beta_{\text{vcenterofquad}} := \beta_{\text{bend}}\left(\alpha_{v1}, \beta_{v1}, K_v, \frac{\text{squad}}{2}\right) \quad \beta_{\text{vcenterofquad}} = 19.8361938 \quad \beta_{v2} := \beta_{\text{vcenterofquad}}$$

$$\alpha_{\text{vcenterofquad}} := \alpha_{\text{bend}}\left(\alpha_{v1}, \beta_{v1}, K_v, \frac{\text{squad}}{2}\right) \quad \alpha_{\text{vcenterofquad}} = 0.0000000 \quad \alpha_{v2} := \alpha_{\text{vcenterofquad}}$$

$$\psi_{\text{vcenterofquad}} := \psi_{\text{bend}}\left(\alpha_{v1}, \beta_{v1}, K_v, \frac{\text{squad}}{2}\right) + \psi_{v1} \quad \psi_{\text{vcenterofquad}} = 0.2888523 \quad \psi_{v2} := \psi_{\text{vcenterofquad}}$$

Horizontal motion.

$$\beta_{\text{hcenterofquad}} := \beta_{\text{bend}}\left(\alpha_{h1}, \beta_{h1}, K_h, \frac{\text{squad}}{2}\right) \quad \beta_{\text{hcenterofquad}} = 7.4028119 \quad \beta_{h2} := \beta_{\text{hcenterofquad}}$$

$$\alpha_{\text{hcenterofquad}} := \alpha_{\text{bend}}\left(\alpha_{h1}, \beta_{h1}, K_h, \frac{\text{squad}}{2}\right) \quad \alpha_{\text{hcenterofquad}} = 0.0000000 \quad \alpha_{h2} := \alpha_{\text{hcenterofquad}}$$

$$\psi_{\text{hcenterofquad}} := \psi_{\text{bend}}\left(\alpha_{h1}, \beta_{h1}, K_h, \frac{\text{squad}}{2}\right) + \psi_{h1} \quad \psi_{\text{hcenterofquad}} = 0.7315061 \quad \psi_{h2} := \psi_{\text{hcenterofquad}}$$

Note that the beta functions are at a maximum (vertical) and a minimum (horizontal) in the center of the quad and that alpha is zero. The amplitude is a maximum in the center of the quad in the vertical and a minimum in the center of the quad in the horizontal. The calculation will show that the variation in the amplitude function is not large.

Next evaluate the CSL parameters at the end of the quad sections. Subscript 1 denotes the value at the beginning of the quad section, and subscript 3 denotes the value at the end of the quad section. See figure above.

$$\begin{aligned}
 \beta_{v3} &:= \beta_{\text{bend}}(\alpha_{v1}, \beta_{v1}, K_v, \text{squad}) & \beta_{v3} &= 19.4258170 \\
 \alpha_{v3} &:= \alpha_{\text{bend}}(\alpha_{v1}, \beta_{v1}, K_v, \text{squad}) & \alpha_{v3} &= 0.1675131 \\
 \psi_{v3} &:= \psi_{\text{bend}}(\alpha_{v1}, \beta_{v1}, K_v, \text{squad}) + \psi_{v1} & \psi_{v3} &= 0.4117326 \\
 \beta_{h3} &:= \beta_{\text{bend}}(\alpha_{h1}, \beta_{h1}, K_h, \text{squad}) & \beta_{h3} &= 7.5967305 \\
 \alpha_{h3} &:= \alpha_{\text{bend}}(\alpha_{h1}, \beta_{h1}, K_h, \text{squad}) & \alpha_{h3} &= -0.0779674 \\
 \psi_{h3} &:= \psi_{\text{bend}}(\alpha_{h1}, \beta_{h1}, K_h, \text{squad}) + \psi_{h1} & \psi_{h3} &= 1.0556343
 \end{aligned}$$

Finally evaluate CLS parameters at the end of the second open section (vertical motion) and second magnet section (horizontal motion). Subscript 3 denotes the value at the beginning of the section, and subscript 4 denotes the value at the end of the section.

$$\begin{aligned}
 \beta_{v4} &:= \beta_{\text{open}}(\alpha_{v3}, \beta_{v3}, \text{sopen}) & \beta_{v4} &= 18.8955947 \\
 \alpha_{v4} &:= \alpha_{\text{open}}(\alpha_{v3}, \beta_{v3}, \text{sopen}) & \alpha_{v4} &= 0.0000000 \\
 \psi_{v4} &:= \psi_{\text{open}}(\alpha_{v3}, \beta_{v3}, \text{sopen}) + \psi_{v3} & \psi_{v4} &= 0.5777047 \\
 \beta_{h4} &:= \beta_{\text{bend}}(\alpha_{h3}, \beta_{h3}, K_{\text{mag}}, \text{smag}) & \beta_{h4} &= 7.8612153 \\
 \alpha_{h4} &:= \alpha_{\text{bend}}(\alpha_{h3}, \beta_{h3}, K_{\text{mag}}, \text{smag}) & \alpha_{h4} &= 0.0000000 \\
 \psi_{h4} &:= \psi_{\text{bend}}(\alpha_{h3}, \beta_{h3}, K_{\text{mag}}, \text{smag}) + \psi_{h3} & \psi_{h4} &= 1.4630121
 \end{aligned}$$

Tabulate in a vector the values of α , β , ψ and verify that that they behave correctly at the end points of sections.

$$\begin{array}{l}
 \text{vertical:} \\
 \text{horizontal:}
 \end{array}
 \begin{array}{l}
 \left(\begin{array}{l} \text{initial_value} \\ \text{entrance_to_quad} \\ \text{center_of_quad} \\ \text{exit_of_quad} \\ \text{final_value} \end{array} \right) \\
 \left(\begin{array}{l} \text{initial_value} \\ \text{entrance_to_quad} \\ \text{center_of_quad} \\ \text{exit_of_quad} \\ \text{final_value} \end{array} \right)
 \end{array}
 \begin{array}{l}
 \alpha_v = \left(\begin{array}{l} 0.0000000 \\ -0.1675131 \\ 0.0000000 \\ 0.1675131 \\ 0.0000000 \end{array} \right) \\
 \alpha_h = \left(\begin{array}{l} 0.0000000 \\ 0.0779674 \\ 0.0000000 \\ -0.0779674 \\ 0.0000000 \end{array} \right)
 \end{array}
 \begin{array}{l}
 \beta_v = \left(\begin{array}{l} 18.8955947 \\ 19.4258170 \\ 19.8361938 \\ 19.4258170 \\ 18.8955947 \end{array} \right) \\
 \beta_h = \left(\begin{array}{l} 7.8612153 \\ 7.5967305 \\ 7.4028119 \\ 7.5967305 \\ 7.8612153 \end{array} \right)
 \end{array}
 \begin{array}{l}
 \psi_v = \left(\begin{array}{l} 0.0000000 \\ 0.1659721 \\ 0.2888523 \\ 0.4117326 \\ 0.5777047 \end{array} \right) \\
 \psi_h = \left(\begin{array}{l} 0.0000000 \\ 0.4073778 \\ 0.7315061 \\ 1.0556343 \\ 1.4630121 \end{array} \right)
 \end{array}$$

Note that the slope, α , is zero initially, in the center of the quad and at the end of the section. Also α has equal magnitudes and opposite signs at the beginning and end of the quad section. Note that the beam envelope, β , is largest (or smallest) in the center of the quad. Now it is possible to calculate the beta function at every point in the first quarter of the ring. Calculate the β function in the three sections using 100 points per section.

$$i := 0, 1.. 100 \quad s_i := \frac{i}{100} \cdot \text{sopen} \quad \text{one step equals } \frac{1}{100} \cdot \text{sopen} = 0.0316526 \text{ meters}$$

$$\text{betav}_i := \beta_{\text{open}}(\alpha_{v0}, \beta_{v0}, s_i) \quad \text{betah}_i := \beta_{\text{bend}}(\alpha_{h0}, \beta_{h0}, K_{\text{mag}}, s_i)$$

$$\text{alphav}_i := \alpha_{\text{open}}(\alpha_{v0}, \beta_{v0}, s_i) \quad \text{alphah}_i := \alpha_{\text{bend}}(\alpha_{h0}, \beta_{h0}, K_{\text{mag}}, s_i)$$

$$\text{psiv}_i := \psi_{\text{open}}(\alpha_{v0}, \beta_{v0}, s_i) \quad \text{psih}_i := \psi_{\text{bend}}(\alpha_{h0}, \beta_{h0}, K_{\text{mag}}, s_i)$$

$$i := 100, 101.. 250 \quad s_i := \frac{i - 100}{150} \cdot \text{squad} + \text{sopen} \quad \text{one step equals } \frac{1}{150} \cdot \text{squad} = 0.0322732 \text{ meters}$$

$$\text{betav}_i := \beta_{\text{bend}}(\alpha_{v1}, \beta_{v1}, K_v, s_i - \text{sopen}) \quad \text{betah}_i := \beta_{\text{bend}}(\alpha_{h1}, \beta_{h1}, K_h, s_i - \text{sopen})$$

$$\text{alphav}_i := \alpha_{\text{bend}}(\alpha_{v1}, \beta_{v1}, K_v, s_i - \text{sopen}) \quad \text{alphah}_i := \alpha_{\text{bend}}(\alpha_{h1}, \beta_{h1}, K_h, s_i - \text{sopen})$$

$$\text{psiv}_i := \psi_{\text{bend}}(\alpha_{v1}, \beta_{v1}, K_v, s_i - \text{sopen}) + \psi_{v1} \quad \text{psih}_i := \psi_{\text{bend}}(\alpha_{h1}, \beta_{h1}, K_h, s_i - \text{sopen}) + \psi_{h1}$$

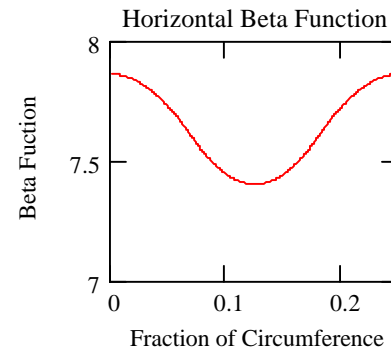
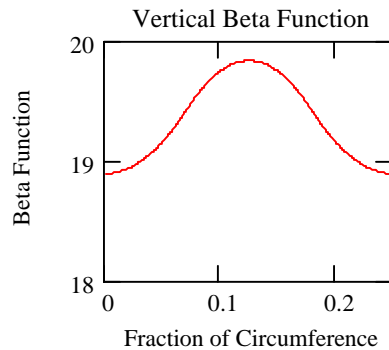
$$i := 250, 251.. 350 \quad s_i := \frac{i - 250}{100} \cdot \text{sopen} + \text{sopen} + \text{squad}$$

$$\text{betav}_i := \beta_{\text{open}}(\alpha_{v3}, \beta_{v3}, s_i - \text{sopen} - \text{squad}) \quad \text{betah}_i := \beta_{\text{bend}}(\alpha_{h3}, \beta_{h3}, K_{\text{mag}}, s_i - \text{sopen} - \text{squad})$$

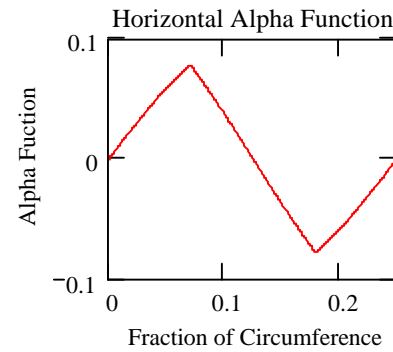
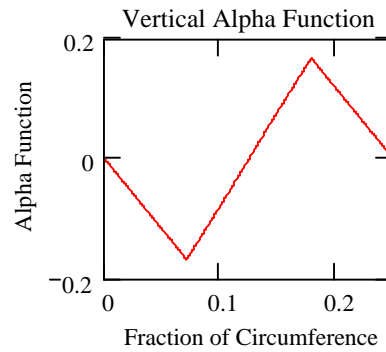
$$\text{alphav}_i := \alpha_{\text{open}}(\alpha_{v3}, \beta_{v3}, s_i - \text{sopen} - \text{squad}) \quad \text{alphah}_i := \alpha_{\text{bend}}(\alpha_{h3}, \beta_{h3}, K_{\text{mag}}, s_i - \text{sopen} - \text{squad})$$

$$\text{psiv}_i := \psi_{\text{open}}(\alpha_{v3}, \beta_{v3}, s_i - \text{sopen} - \text{squad}) + \psi_{v3} \quad \text{psih}_i := \psi_{\text{bend}}(\alpha_{h3}, \beta_{h3}, K_{\text{mag}}, s_i - \text{sopen} - \text{squad}) + \psi_{h3}$$

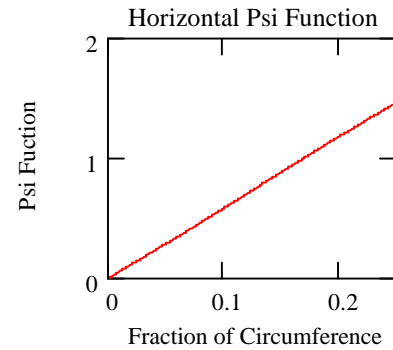
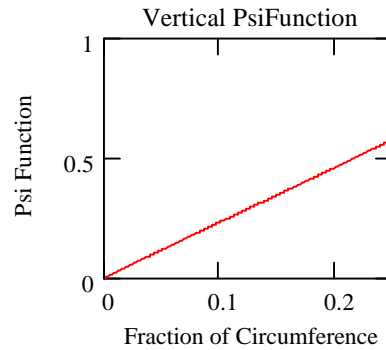
Plot the beta functions for one period. Note the offsets in the vertical scales.



Note that the variation in the beta functions is small. Also plot alpha functions.



Note that the alpha functions are small. Also plot psi functions.



Note that the psi functions are smooth.

Just for completeness plot the beta and psi functions for the entire ring. These are useful when calculating trajectories of particles.

$i := 351, 352..700$ $s_i := s_{i-350} + 2 \cdot s_{open} + s_{quad}$

$betav_i := betav_{i-350}$ $betah_i := betah_{i-350}$ $psiv_i := psiv_{i-350} + psiv_{350}$ $psih_i := psih_{i-350} + psih_{350}$

$alphav_i := alphav_{i-350}$ $alphah_i := alphah_{i-350}$

$i := 701, 702..1050$ $s_i := s_{i-700} + 4 \cdot s_{open} + 2 \cdot s_{quad}$

$betav_i := betav_{i-700}$ $betah_i := betah_{i-700}$ $psiv_i := psiv_{i-700} + psiv_{700}$ $psih_i := psih_{i-700} + psih_{700}$

$alphav_i := alphav_{i-700}$ $alphah_i := alphah_{i-700}$

$i := 1051, 1052..139$; $s_i := s_{i-1050} + 6 \cdot \text{sopen} + 3 \cdot \text{squad}$

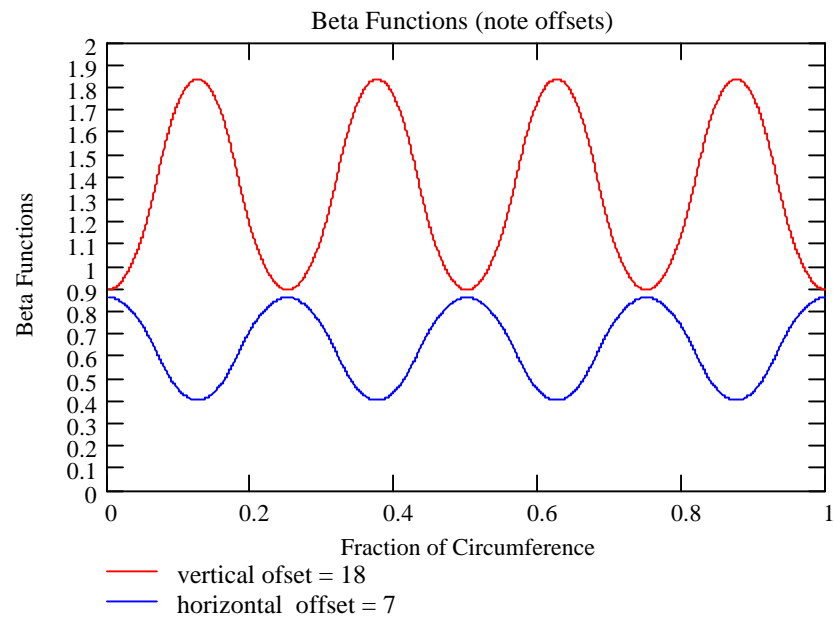
$\text{betav}_i := \text{betav}_{i-1050}$ $\text{betah}_i := \text{betah}_{i-1050}$ $\text{psiv}_i := \text{psiv}_{i-1050} + \text{psiv}_{1050}$ $\text{psih}_i := \text{psih}_{i-1050} + \text{psih}_{1050}$

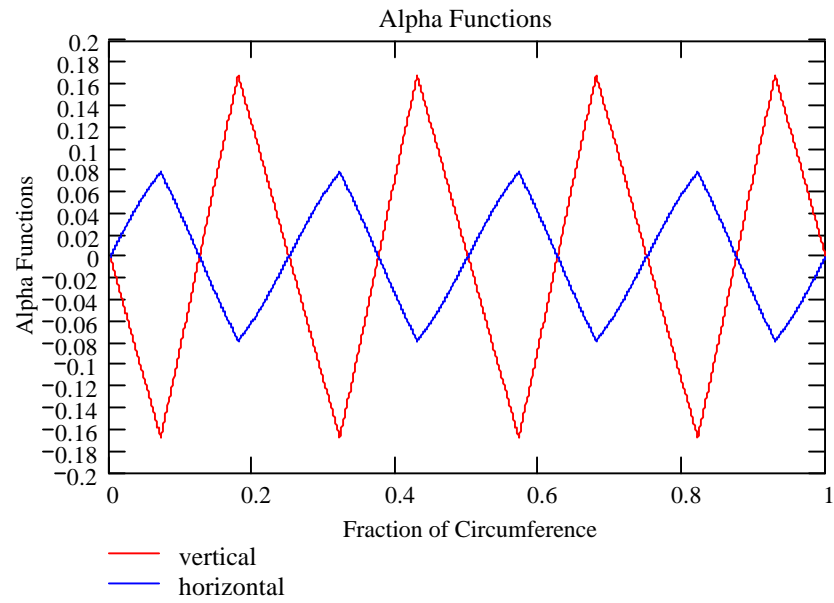
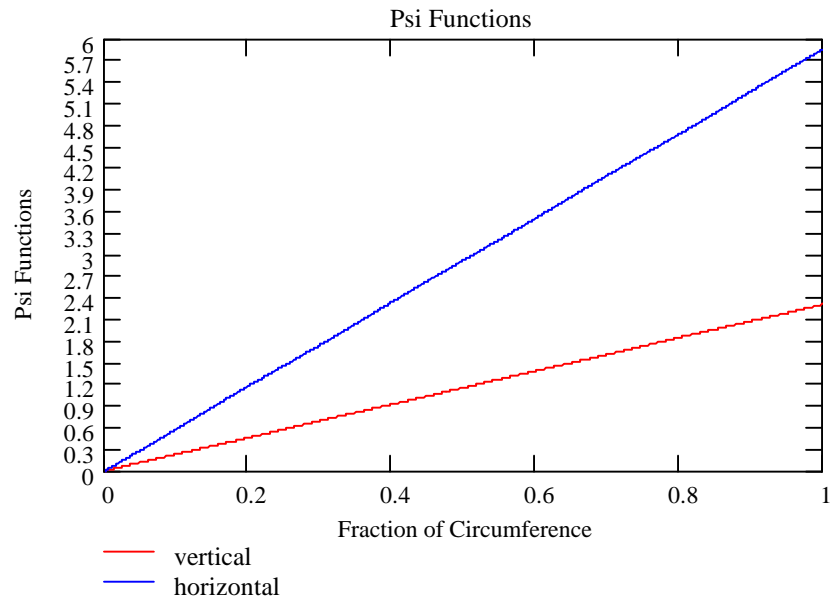
$\text{alphav}_i := \text{alphav}_{i-1050}$ $\text{alphah}_i := \text{alphah}_{i-1050}$

$\text{betav}_{1400} := \text{betav}_0$ $\text{betah}_{1400} := \text{betah}_0$ $\text{psiv}_{1400} := \mu v$ $\text{psih}_{1400} := \mu h \text{prime}$ $s_{1400} := 8 \cdot \text{sopen} + 4 \cdot \text{squad}$

$\text{alphav}_{1400} := \text{alphav}_0$ $\text{alphah}_{1400} := \text{alphah}_0$

Note offsets in the vertical scale.





Calculate the tune using $\mu = \psi(L_{v,h}) - \psi(0) = \int_0^{L_{v,h}} \frac{1}{\beta} ds$. Since beta function is so smooth just do simple sum.

vertical tune

$$v_{vopen} := \frac{sopen}{100 \cdot 2 \cdot \pi} \cdot \left(\frac{1}{2} \cdot \frac{1}{betav_0} + \sum_{i=1}^{99} \frac{1}{betav_i} + \frac{1}{2} \cdot \frac{1}{betav_{100}} \right) \quad v_{vquad} := \frac{squad}{150 \cdot 2 \cdot \pi} \cdot \left(\frac{1}{2} \cdot \frac{1}{betav_{100}} + \sum_{i=1}^{149} \frac{1}{betav_{i+100}} + \frac{1}{2} \cdot \frac{1}{betav_{250}} \right)$$

$$v_v := 4 \cdot (2 \cdot v_{vopen} + v_{vquad}) \quad v_v = 0.3677782$$

We also found the tune from CS matrix trace, equation 2.20. $\frac{\mu_v}{2 \cdot \pi} = 0.3677782$ and we found the tune from calculating $\psi \frac{psiv_{1400}}{2 \cdot \pi} = 0.3677782$

Integration of beta functions gives the same result as CLS matrix trace.

Vertical betatron wavelength then is $\frac{1}{v_v} = 2.7190299$ times $2\pi \cdot R$, or $2\pi \cdot R \cdot \frac{1}{v_v} = 121.5026067$ m $\lambda_v := 2\pi \cdot R \cdot \frac{1}{v_v} \quad \lambda_v = 121.5026067$

horizontal tune

$$v_{hopen} := \frac{sopen}{100 \cdot 2 \cdot \pi} \cdot \left(\frac{1}{2} \cdot \frac{1}{betah_0} + \sum_{i=1}^{99} \frac{1}{betah_i} + \frac{1}{2} \cdot \frac{1}{betah_{100}} \right) \quad v_{hquad} := \frac{squad}{150 \cdot 2 \cdot \pi} \cdot \left(\frac{1}{2} \cdot \frac{1}{betah_{100}} + \sum_{i=1}^{149} \frac{1}{betah_{i+100}} + \frac{1}{2} \cdot \frac{1}{betah_{250}} \right)$$

$$v_h := 4 \cdot (2 \cdot v_{hopen} + v_{hquad}) \quad v_h = 0.9313824$$

We also found the tune from CS matrix trace, equation 2.20 $\frac{\mu_{hprime}}{2 \cdot \pi} = 0.9313825$ and we found the tune from calculating $\psi \frac{psih_{1400}}{2 \cdot \pi} = 0.9313825$

Integration of beta functions gives the same result as CLS matrix trace.

Horizontal betatron wavelength then is $\frac{1}{v_h} = 1.0736728$ times $2\pi \cdot R$, or $2\pi \cdot R \cdot \frac{1}{v_h} = 47.9781576$ m $\lambda_h := 2\pi \cdot R \cdot \frac{1}{v_h} \quad \lambda_h = 47.9781576$

Compare again calculated tune to tune with complete quad coverage.

vertical tune is $\sqrt{n_{average}} = 0.3661038$ and vertical betatron wavelength is $\frac{1}{\sqrt{n_{average}}} = 2.7314657$ times $2\pi \cdot R \cdot \frac{1}{\sqrt{n_{average}}} = 122.0583142$ m

horizontal tune is $\sqrt{1 - n_{average}} = 0.9305740$ horizontal betatron wavelength is $\frac{1}{\sqrt{1 - n_{average}}} = 1.0746055$ times $2\pi \cdot R \cdot \frac{1}{\sqrt{1 - n_{average}}} = 48.0198384$ m

The calculation using naverage is excellent $\frac{v_v - \sqrt{\text{naverage}}}{v_v} = 0.0045528$ $\frac{v_h - \sqrt{1 - \text{naverage}}}{v_h} = 0.0008680$

Calculate cyclotron frequency. We need velocity.

Use muon anomaly to calculate magic gamma. $\gamma_{\text{magic}} := \sqrt{1 + \frac{1}{a}}$ $\gamma_{\text{magic}} = 29.3034394$ $\beta_{\text{magic}} := \frac{\sqrt{\gamma_{\text{magic}}^2 - 1}}{\gamma_{\text{magic}}}$ $\beta_{\text{magic}} = 0.9994175$

Cyclotron frequency: $f_{\text{cyclotron}} := \frac{c \cdot \beta_{\text{magic}}}{2 \cdot \pi \cdot R}$, or $f_{\text{cyclotron}} = 6.7049579 \times 10^6$ Hz. Period is $T_{\text{cyclotron}} := \frac{1}{f_{\text{cyclotron}}}$, or $T_{\text{cyclotron}} = 1.4914337 \times 10^{-7}$ s.

Calculate betatron frequency

Vertical betatron frequency: $\frac{c \cdot \beta_{\text{magic}}}{\lambda_v} = 2.4659376 \times 10^6$ Hz vertical betatron period is $\frac{1}{\left(\frac{c \cdot \beta_{\text{magic}}}{\lambda_v}\right)} = 4.0552527 \times 10^{-7}$ s

horizontal betatron frequency: $\frac{c \cdot \beta_{\text{magic}}}{\lambda_h} = 6.2448801 \times 10^6$ Hz horizontal betatron period is $\frac{1}{\left(\frac{c \cdot \beta_{\text{magic}}}{\lambda_h}\right)} = 1.6013118 \times 10^{-7}$ s. Define

$$f_{\text{vbeta}} := \frac{c \cdot \beta_{\text{magic}}}{\lambda_v} \quad f_{\text{hbeta}} := \frac{c \cdot \beta_{\text{magic}}}{\lambda_h}$$

Compare frequencies to frequencies with complete quad coverage.

Vertical betatron frequency then is $f_v := \frac{c \cdot \beta_{\text{magic}}}{2 \cdot \pi \cdot R} \cdot \sqrt{\text{naverage}}$, or $f_v = 2.4547106 \times 10^6$ Hz vertical betatron period is $\frac{1}{f_v} = 4.0737999 \times 10^{-7}$ s

$$\frac{f_{\text{vbeta}} - f_v}{f_{\text{vbeta}}} = 0.4552803\%$$

Horizontal betatron frequency then is $f_h := \frac{c \cdot \beta_{\text{magic}}}{2 \cdot \pi \cdot R} \cdot \sqrt{1 - \text{naverage}}$, or $f_h = 6.2394596 \times 10^6$ Hz horizontal betatron period is $\frac{1}{f_h} = 1.6027029 \times 10^{-7}$ s

$$\frac{f_{\text{hbeta}} - f_h}{f_{\text{hbeta}}} = 0.0867991\%$$

CBO frequency related to n value $f_{\text{hCBO}} := \frac{c \cdot \beta_{\text{magic}}}{2 \cdot \pi \cdot R} \cdot (1 - \sqrt{1 - \text{naverage}})$ $1 - \left[1 - \left(465.5 \cdot 10^3 \cdot \frac{2\pi \cdot R}{c \cdot \beta_{\text{magic}}} \right) \right]^2 = 0.1340325$

$$f_{\text{hCBO}} = 4.6549830 \times 10^5$$

The simple calculation of frequencies is clearly very good; frequency from exact tune differs by one or five parts in 1,000

Write out array with s, alphah, betah, psih, alphav, betav, psiv in order of IDA program (see below).

m := 0, 1.. 1400

output_{m,0} := s_m output_{m,1} := alphah_m output_{m,2} := betah_m output_{m,3} := psih_m output_{m,4} := alphav_m output_{m,5} := betav_m output_{m,6} := psiv_m

PRNCOLWIDTH:= 10 PRNPRECISION:= 10

WRITEPRN("D:\Accelerator_physics\G-2_ring\results.txt") := output

Recall general form of solution and slope. We have an array with 1400 values of s, alpha, beta and psi. Normalization A and phase delta are given by initial conditions for particle.

$$y(s) = \sqrt{A\beta(s)} \cdot \cos(\psi(s) + \delta)$$

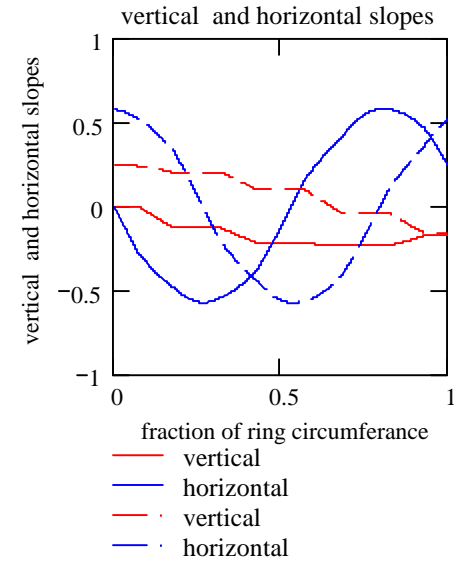
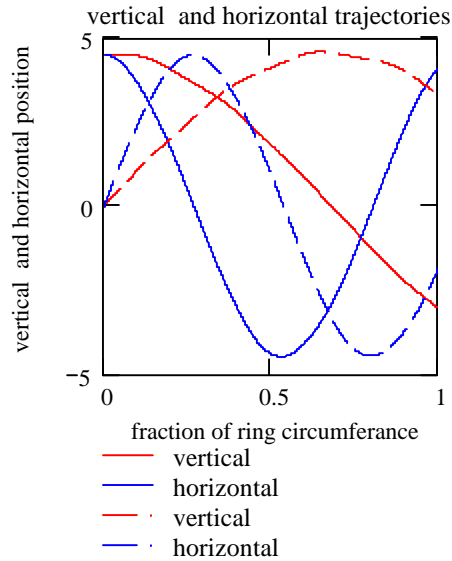
$$y'(s) = -\sqrt{\frac{A}{\beta(s)}} (\alpha(s) \cdot \cos(\psi(s) + \delta) + \sin(\psi(s) + \delta))$$

Plot horizontal and vertical motion for two choices of amplitude and phase corresponding to zero slope (cosine like) and zero displacement (sine like) at s = 0. Also calculate and plot horizontal and vertical slope. Note that slope obtained from differentiation of position.

cosine like trajectory	$A_{\text{normc}} := \left(\frac{4.5}{\sqrt{\beta h_0}}\right)^2$	$A_{\text{normc}} := \left(\frac{4.5}{\sqrt{\beta v_0}}\right)^2$	$\delta_{\text{hc}} := 0$	$\delta_{\text{vc}} := 0$
sine like trajectory	$A_{\text{norms}} := \left(\frac{4.5}{\sqrt{\beta h_0}}\right)^2$	$A_{\text{norms}} := \left(\frac{4.5}{\sqrt{\beta v_0}}\right)^2$	$\delta_{\text{hs}} := \frac{-\pi}{2}$	$\delta_{\text{vs}} := \frac{-\pi}{2}$

$x_{\text{c}_m} := \sqrt{A_{\text{normc}} \cdot \text{betah}_m} \cdot \cos(\text{psih}_m + \delta_{\text{hc}})$	$z_{\text{c}_m} := \sqrt{A_{\text{normc}} \cdot \text{betav}_m} \cdot \cos(\text{psiv}_m + \delta_{\text{vc}})$
$x_{\text{s}_m} := \sqrt{A_{\text{norms}} \cdot \text{betah}_m} \cdot \cos(\text{psih}_m + \delta_{\text{hs}})$	$z_{\text{s}_m} := \sqrt{A_{\text{norms}} \cdot \text{betav}_m} \cdot \cos(\text{psiv}_m + \delta_{\text{vs}})$

$x_{\text{cp}_m} := -1 \cdot \sqrt{\frac{A_{\text{normc}}}{\text{betah}_m}} \cdot (\text{alphah}_m \cdot \cos(\text{psih}_m + \delta_{\text{hc}}) + \sin(\text{psih}_m + \delta_{\text{hc}}))$	$x_{\text{sp}_m} := -1 \cdot \sqrt{\frac{A_{\text{norms}}}{\text{betah}_m}} \cdot (\text{alphah}_m \cdot \cos(\text{psih}_m + \delta_{\text{hs}}) + \sin(\text{psih}_m + \delta_{\text{hs}}))$
$z_{\text{cp}_m} := -1 \cdot \sqrt{\frac{A_{\text{normc}}}{\text{betav}_m}} \cdot (\text{alphav}_m \cdot \cos(\text{psiv}_m + \delta_{\text{vc}}) + \sin(\text{psiv}_m + \delta_{\text{vc}}))$	$z_{\text{sp}_m} := -1 \cdot \sqrt{\frac{A_{\text{norms}}}{\text{betav}_m}} \cdot (\text{alphav}_m \cdot \cos(\text{psiv}_m + \delta_{\text{vs}}) + \sin(\text{psiv}_m + \delta_{\text{vs}}))$



Solid lines are cosine-like trajectories. Dashed lines are sine-like trajectories.

We can also specify initial position and slope and calculate amplitude A and phase δ .

$$y(s) = \sqrt{A\beta(s)} \cdot \cos(\psi(s) + \delta)$$

$$y'(s) = -\sqrt{\frac{A}{\beta(s)}} \cdot (\alpha(s) \cdot \cos(\psi(s) + \delta) + \sin(\psi(s) + \delta))$$

$$y_o = \sqrt{A\beta_o} \cdot \cos(\psi_o + \delta)$$

$$y'_o = -\sqrt{\frac{A}{\beta_o}} \cdot (\alpha_o \cos(\psi_o + \delta) + \sin(\psi_o + \delta))$$

$$A = y_o^2 \frac{1 + \alpha_o^2}{\beta_o} + y_o'^2 \beta_o + 2\alpha_o y_o y_o'$$

$$\delta = \tan^{-1}\left(\beta_o \frac{y_o'}{y_o} + \alpha_o\right) - \psi_o$$

$$\text{Amph}(x, xp) := x^2 \cdot \frac{1 + (\text{alphah}_0)^2}{\text{betah}_0} + \text{betah}_0 \cdot xp^2 + 2 \cdot \text{alphah}_0 \cdot x \cdot xp \quad \text{deltah}(x, xp) := \text{atan} \left(\text{betah}_0 \cdot \frac{xp}{x} + \text{alphah}_0 \right) - \text{psih}_0$$

$$\text{Ampv}(z, zp) := z^2 \cdot \frac{1 + (\text{alphav}_0)^2}{\text{betav}_0} + \text{betav}_0 \cdot zp^2 + 2 \cdot \text{alphav}_0 \cdot z \cdot zp \quad \text{deltav}(z, zp) := \text{atan} \left(\text{betav}_0 \cdot \frac{zp}{z} + \text{alphav}_0 \right) - \text{psiv}_0$$

Note that these expressions could simplify considerably since at the initial position, 0, alpha and psi are 0.

Verify that these functions reproduce amplitudes and phases found above. Initial position and slopes are

$$xc_0 = 4.5000000 \quad xcp_0 = 0.0000000 \quad xs_0 = 0.0000000 \quad xsp_0 = 0.5724306 \quad zc_0 = 4.5000000 \quad zcp_0 = 0.0000000 \quad zs_0 = 0.0000000 \quad zsp_0 = 0.2381507$$

Then amplitudes and phases are

$$\begin{aligned} \text{Amph}(xc_0, xcp_0) &= 2.5759376 & \text{Ahnormc} &= 2.5759376 \\ \text{deltah}(xc_0, xcp_0) &= 0.0000000 & \delta_{hc} &= 0.0000000 \\ \text{Ampv}(zc_0, zcp_0) &= 1.0716784 & \text{Avnormc} &= 1.0716784 \\ \text{deltav}(zc_0, zcp_0) &= 0.0000000 & \delta_{vc} &= 0.0000000 \\ \text{Amph}(xs_0, xsp_0) &= 2.5759376 & \text{Ahnorms} &= 2.5759376 \\ \text{deltah}(xs_0, xsp_0) &= 1.5707963 & \delta_{hs} &= -1.5707963 \\ \text{Ampv}(zs_0, zsp_0) &= 1.0716784 & \text{Avnorms} &= 1.0716784 \\ \text{deltav}(zs_0, zsp_0) &= 1.5707963 & \delta_{vs} &= -1.5707963 \end{aligned}$$

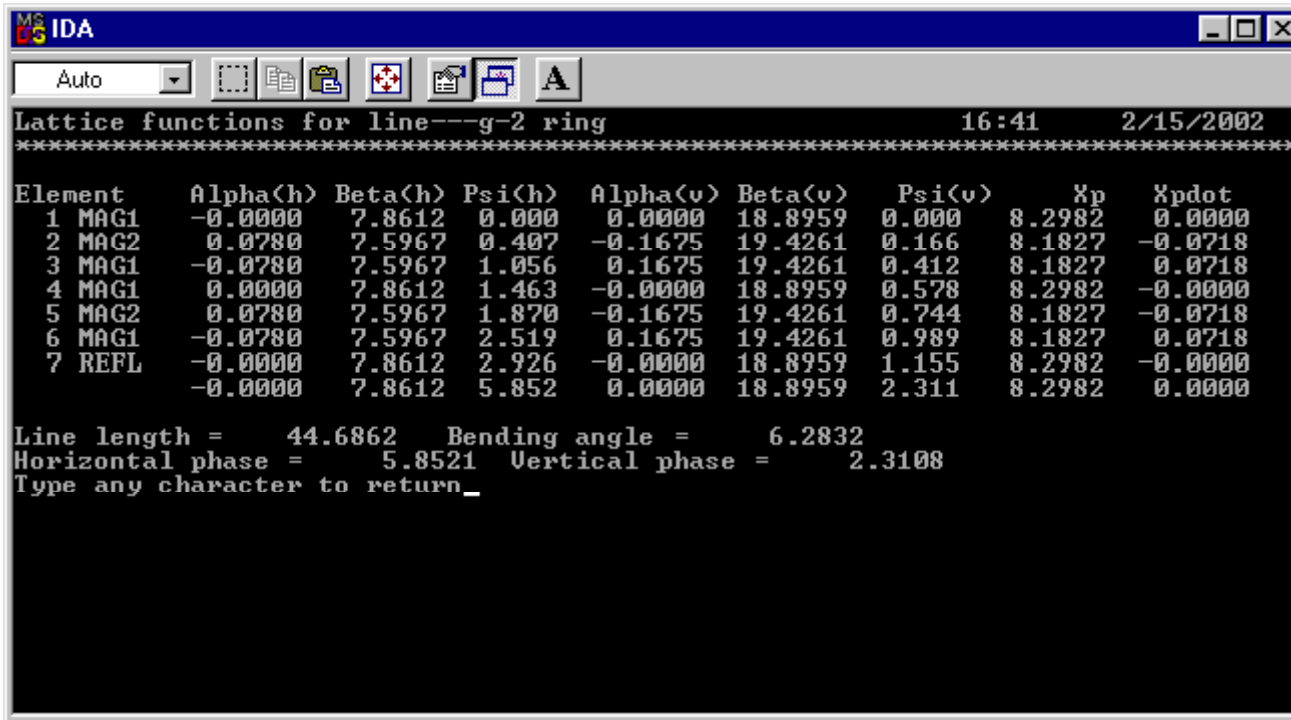
Thus the functions are correct.

 Program IDA, written by Mark Barton of BNL, will calculate a lattice. Next figure is input file for IDA describing g-2 ring.

```

MS IDA
Auto
E<dit,F<it,G<et,S<ave,W<rite,T<ype,M<ult,C<ycle,I<ntegrals,P<lot,Q<uit
g-2 ring of BNL E821
Parameters for g-2 ring in pathway
Number of elements = 7      linelength = 44.686
Name      Length      Grad      Rho      Ent. angle      Exit angle
1 MAG1    3.1653    0.000000    7.1120    0.0000    0.0000
2 MAG2    4.8409   -0.006115    7.1120    0.0000    0.0000
3 MAG1    3.1653    0.000000    7.1120    0.0000    0.0000
4 MAG1    3.1653    0.000000    7.1120    0.0000    0.0000
5 MAG2    4.8409   -0.006115    7.1120    0.0000    0.0000
6 MAG1    3.1653    0.000000    7.1120    0.0000    0.0000
7 REFL
  
```

This input describes two periods of the ring. MAG1 is the non-quad section. MAG2 is the quad section. REFL repeats the elements and thus specifying the entire ring. MAG1 only has bending. MAG2 also has a gradient equal to n/R^2 . Units are meters. Next figure is output file from IDA.



The values of alpha, beta and psi shown by IDA are the values at the beginning of the section. Thus the line for the first MAG1 gives the values at position 0 (initial); the line for the first MAG2 gives the values at position 1 (entrance to the quad). The line for the second MAG1 gives the values at position 3 (end of the quad), etc. My Mathcad calculation agrees with IDA output. Recall from above. In the IDA output Xp and Xpdot are the dispersion functions which are not treated in this note.

$$\begin{array}{l}
 \text{horizontal} \\
 \text{vertical:}
 \end{array}
 \begin{array}{l}
 \left(\begin{array}{l} \text{initial_value} \\ \text{entrance_to_quad} \\ \text{center_of_quad} \\ \text{exit_of_quad} \\ \text{final_value} \end{array} \right) \\
 \left(\begin{array}{l} \text{initial_value} \\ \text{entrance_to_quad} \\ \text{center_of_quad} \\ \text{exit_of_quad} \\ \text{final_value} \end{array} \right)
 \end{array}
 \begin{array}{l}
 \alpha_h = \left(\begin{array}{l} 0.0000000 \\ 0.0779674 \\ 0.0000000 \\ -0.0779674 \\ 0.0000000 \end{array} \right) \\
 \alpha_v = \left(\begin{array}{l} 0.0000000 \\ -0.1675131 \\ 0.0000000 \\ 0.1675131 \\ 0.0000000 \end{array} \right)
 \end{array}
 \begin{array}{l}
 \beta_h = \left(\begin{array}{l} 7.8612153 \\ 7.5967305 \\ 7.4028119 \\ 7.5967305 \\ 7.8612153 \end{array} \right) \\
 \beta_v = \left(\begin{array}{l} 18.8955947 \\ 19.4258170 \\ 19.8361938 \\ 19.4258170 \\ 18.8955947 \end{array} \right)
 \end{array}
 \begin{array}{l}
 \psi_h = \left(\begin{array}{l} 0.0000000 \\ 0.4073778 \\ 0.7315061 \\ 1.0556343 \\ 1.4630121 \end{array} \right) \\
 \psi_v = \left(\begin{array}{l} 0.0000000 \\ 0.1659721 \\ 0.2888523 \\ 0.4117326 \\ 0.5777047 \end{array} \right)
 \end{array}$$

Now compare to Design Report Figure 5.2.2. Qualitatively, the plots are the same. Quantitatively, the values of the beta functions are different. In particular, in the Design Report the vertical beta functions oscillates between 21 and 22. In my calculation it oscillates between 18.9 and 19.4. The horizontal beta function does appear to have the same values in the Design Report and my calculation. The Design Report is not sufficiently well document to identify the source of the difference.

Variation in beam size in the entire ring is small. Recall that root of the β function is proportional to beam size. Then the variation in the beam size is given by the square root of the ratio of the beta functions at the center of the quad and at the beginning of the open section.

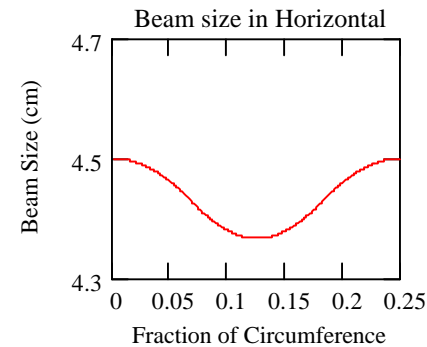
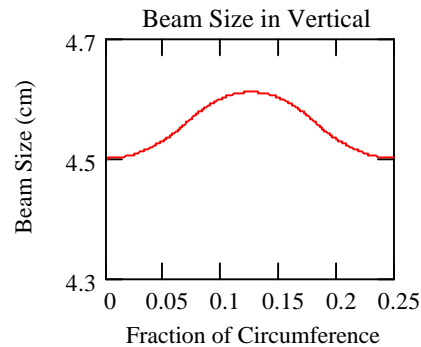
$$\sqrt{\frac{\beta_{\text{centerofquad}}}{\beta_{v0}}} = 1.0245871 \quad \sqrt{\frac{\beta_{h0}}{\beta_{\text{centerofquad}}}} = 1.0304964$$

The variation is a few percent.

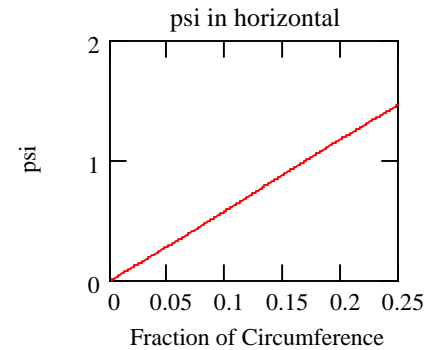
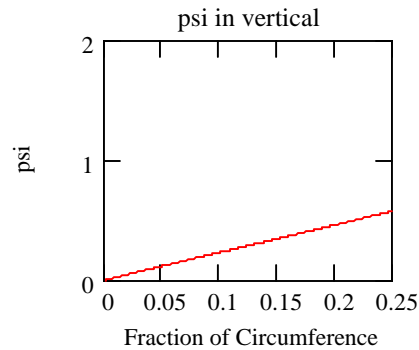
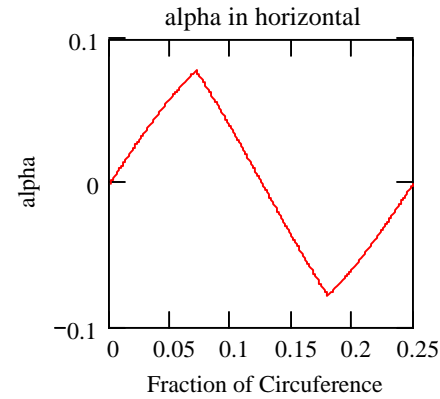
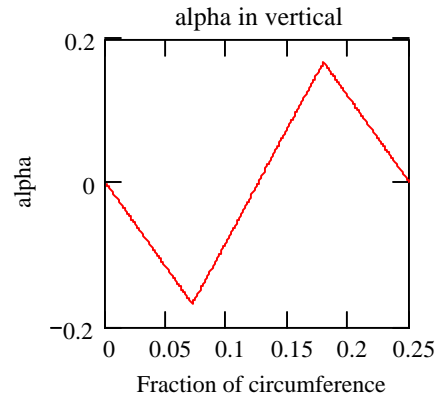
Recall that equation of motion for particle is $y(s) = \sqrt{A \cdot \beta(s)} \cdot \cos(\psi(s) + \delta)$. Collimators in bellows constrain the beam to 4.5 cm in radius. If a collimator is placed at beginning and end of O(M) sections, beam must be no more than 4.5 cm there. Make maximum amplitude there 4.5 cm by appropriate choice of A and plot the beam size.

$$A_{\text{vnorm}} := \left(\frac{4.5}{\sqrt{\beta_{v0}}} \right)^2 \quad A_{\text{hnorm}} := \left(\frac{4.5}{\sqrt{\beta_{h0}}} \right)^2$$

$$i := 0, 1..350 \quad \text{amplitude}_{h_i} := \sqrt{A_{\text{hnorm}} \cdot \beta_{h_i}} \quad \text{amplitude}_{v_i} := \sqrt{A_{\text{vnorm}} \cdot \beta_{v_i}}$$



Also plot α and ψ functions.



Calculate vertical and horizontal tunes from psi (just to check plots).

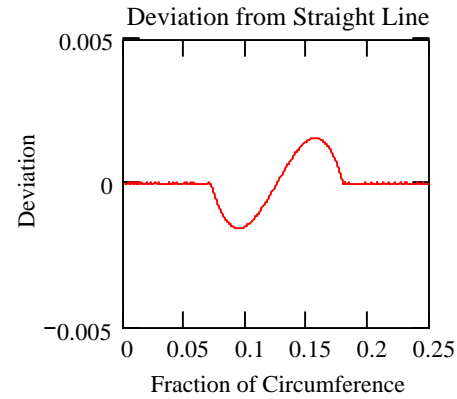
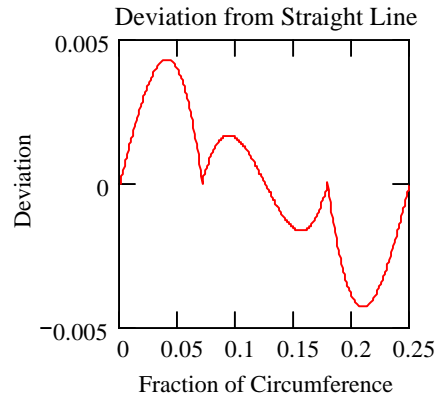
$$v_v := \frac{4}{2 \cdot \pi} \cdot \text{psiv}_{350} \quad v_v = 0.3677782 \quad v_h := \frac{4}{2 \cdot \pi} \cdot \text{psih}_{350} \quad v_h = 0.9313825 \quad (v_v)^2 + (v_h)^2 = 1.0027341$$

Note that sum of squares of the tunes is a little larger than 1, since we do not have continuous quads and thus a classic weak focusing ring.

The alpha function appears to be straight line segments, but it does have a sinusoidal component in the bends. Calculate straight line segments between the end points of the sections.

$$\begin{aligned} i := 0, 1..99 \quad \text{alphavstraight}_i &:= \frac{\alpha v_1 - \alpha v_0}{\text{sopen}} \cdot \frac{i}{100} \cdot \text{sopen} + \alpha v_0 & \text{alphahstraight}_i &:= \frac{\alpha h_1 - \alpha h_0}{\text{sopen}} \cdot \frac{i}{100} \cdot \text{sopen} + \alpha h_0 \\ i := 100, 101..249 \quad \text{alphavstraight}_i &:= \frac{\alpha v_3 - \alpha v_1}{\text{squad}} \cdot \frac{i - 100}{150} \cdot \text{squad} + \alpha v_1 & \text{alphahstraight}_i &:= \frac{\alpha h_3 - \alpha h_1}{\text{squad}} \cdot \frac{i - 100}{150} \cdot \text{squad} + \alpha h_1 \\ i := 250, 251..350 \quad \text{alphavstraight}_i &:= \frac{\alpha v_4 - \alpha v_3}{\text{sopen}} \cdot \frac{i - 250}{100} \cdot \text{sopen} + \alpha v_3 & \text{alphahstraight}_i &:= \frac{\alpha h_4 - \alpha h_3}{\text{sopen}} \cdot \frac{i - 250}{100} \cdot \text{sopen} + \alpha h_3 \end{aligned}$$

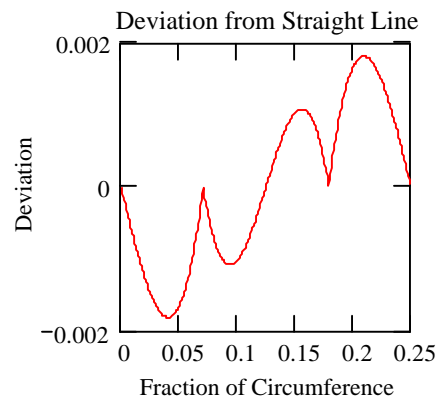
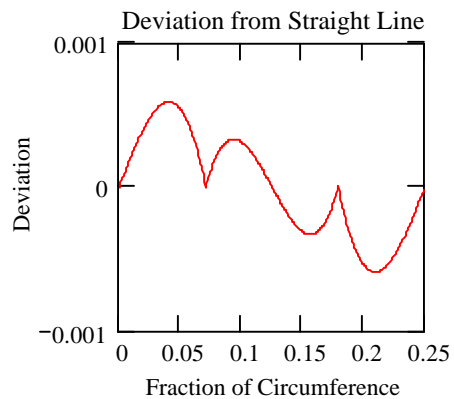
Plot the deviation of alpha function from straight line. Vertical on left, horizontal on right.



Similarly psi function appears to be a straight line. Again calculate straight line segments between the end points of the sections.

$$\begin{aligned}
 i := 0, 1..99 \quad \text{psivstraight}_i &:= \frac{\psi^v_1 - \psi^v_0}{\text{sopen}} \cdot \frac{i}{100} \cdot \text{sopen} + \psi^v_0 & \text{psihstraight}_i &:= \frac{\psi^h_1 - \psi^h_0}{\text{sopen}} \cdot \frac{i}{100} \cdot \text{sopen} + \psi^h_0 \\
 i := 100, 101..249 \quad \text{psivstraight}_i &:= \frac{\psi^v_3 - \psi^v_1}{\text{squad}} \cdot \frac{i - 100}{150} \cdot \text{squad} + \psi^v_1 & \text{psihstraight}_i &:= \frac{\psi^h_3 - \psi^h_1}{\text{squad}} \cdot \frac{i - 100}{150} \cdot \text{squad} + \psi^h_1 \\
 i := 250, 251..350 \quad \text{psivstraight}_i &:= \frac{\psi^v_4 - \psi^v_3}{\text{sopen}} \cdot \frac{i - 250}{100} \cdot \text{sopen} + \psi^v_3 & \text{psihstraight}_i &:= \frac{\psi^h_4 - \psi^h_3}{\text{sopen}} \cdot \frac{i - 250}{100} \cdot \text{sopen} + \psi^h_3
 \end{aligned}$$

Plot the deviation of psi function from a straight line. Vertical on left, horizontal on right.



How to propagate a particle beyond one turn? We have single turn transport matrix with initial position at the middle of an open section. Evaluate transport matrices from alpha, beta, and psi and compare to Lh and Lv.

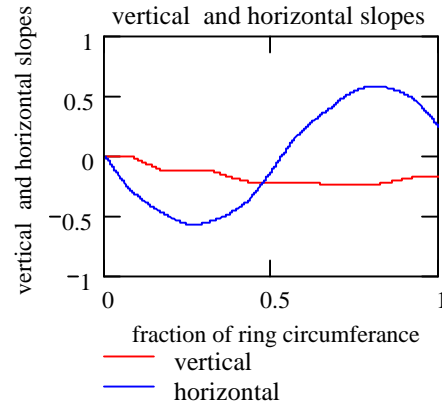
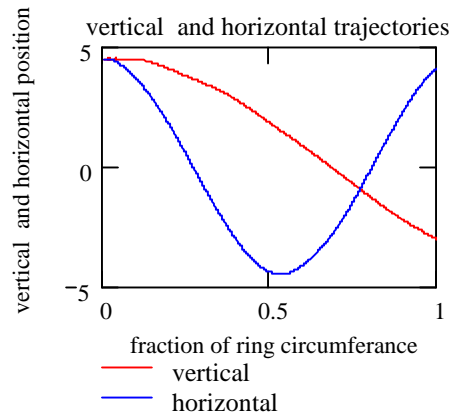
$$L_h = \begin{pmatrix} 0.9084913 & -3.2852315 \\ 0.0531602 & 0.9084913 \end{pmatrix} \begin{bmatrix} \cos(\psi_{1400}) + \alpha_{1400} \cdot \sin(\psi_{1400}) & \beta_{1400} \cdot \sin(\psi_{1400}) \\ -1 \cdot \frac{1 + (\alpha_{1400})^2}{\beta_{1400}} \cdot \sin(\psi_{1400}) & \cos(\psi_{1400}) - \alpha_{1400} \cdot \sin(\psi_{1400}) \end{bmatrix} = \begin{pmatrix} 0.9084913 & -3.2852315 \\ 0.0531602 & 0.9084913 \end{pmatrix}$$

$$L_v = \begin{pmatrix} -0.6743045 & 13.9535167 \\ -0.0390807 & -0.6743045 \end{pmatrix} \begin{bmatrix} \cos(\psi_{1400}) + \alpha_{1400} \cdot \sin(\psi_{1400}) & \beta_{1400} \cdot \sin(\psi_{1400}) \\ -1 \cdot \frac{1 + (\alpha_{1400})^2}{\beta_{1400}} \cdot \sin(\psi_{1400}) & \cos(\psi_{1400}) - \alpha_{1400} \cdot \sin(\psi_{1400}) \end{bmatrix} = \begin{pmatrix} -0.6743045 & 13.9535167 \\ -0.0390807 & -0.6743045 \end{pmatrix}$$

The matrices agree.

Recall from above that we have obtained arrays s, alphah, betah, psih, alphav, betav, psiv, and we have calculated horizontal and vertical trajectories and slopes, x, xp, z, and zp, as a function of s for cosine and sine like trajectories. These arrays are evaluated for index of 0 to 1400.

Copy graphs from above for cosine like trajectories.



Initial value of trajectories

$$x_{c_0} = 4.5000000 \quad z_{c_0} = 4.5000000 \quad x_{cp_0} = 0.0000000 \quad z_{cp_0} = 0.0000000$$

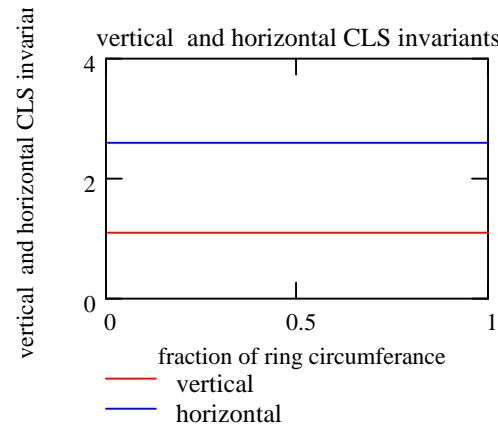
Final value of trajectories

$$x_{c_{1400}} = 4.0882108 \quad z_{c_{1400}} = -3.0343702 \quad x_{cp_{1400}} = 0.2392209 \quad z_{cp_{1400}} = -0.1758632$$

We do get single turn transport. Compare arrays x, xp, z, zp to single turn transport. $L_h \begin{pmatrix} x_{c0} \\ x_{cp0} \end{pmatrix} = \begin{pmatrix} 4.0882108 \\ 0.2392209 \end{pmatrix}$ $L_v \begin{pmatrix} z_{c0} \\ z_{cp0} \end{pmatrix} = \begin{pmatrix} -3.0343702 \\ -0.1758632 \end{pmatrix}$

Evaluate and plot horizontal and vertical CLS invariant amplitudes. We should obtain a constant, and in our case Ahnorm and Avnorm.

$$i := 0, 1..140 \quad CLSh_i := \frac{1 + (\alpha_{hi})^2}{\beta_{hi}} \cdot (x_{ci})^2 + 2 \cdot \alpha_{hi} \cdot x_{ci} \cdot x_{cpi} + \beta_{hi} \cdot (x_{cpi})^2 \quad CLSv_i := \frac{1 + (\alpha_{vi})^2}{\beta_{vi}} \cdot (z_{ci})^2 + 2 \cdot \alpha_{vi} \cdot z_{ci} \cdot z_{cpi} + \beta_{vi} \cdot (z_{cpi})^2$$



$$Ahnorm = 2.5759376 \quad Avnorm = 1.0716784$$

We can propagate particle around the ring and evaluate position and slope at a given position just from the single turn transport matrix.

$$i := 0, 1..100 \quad \begin{pmatrix} x_{turn_0} \\ x_{pturn_0} \end{pmatrix} := \begin{pmatrix} x_{c0} \\ x_{cp0} \end{pmatrix} \quad \begin{pmatrix} z_{turn_0} \\ z_{pturn_0} \end{pmatrix} := \begin{pmatrix} z_{c0} \\ z_{cp0} \end{pmatrix} \quad \begin{pmatrix} x_{turn_{i+1}} \\ x_{pturn_{i+1}} \end{pmatrix} := L_h \cdot \begin{pmatrix} x_{turn_i} \\ x_{pturn_i} \end{pmatrix} \quad \begin{pmatrix} z_{turn_{i+1}} \\ z_{pturn_{i+1}} \end{pmatrix} := L_v \cdot \begin{pmatrix} z_{turn_i} \\ z_{pturn_i} \end{pmatrix}$$

Particle position at center of open section for 100 turns

